An Analysis of Two-Impulse Orbital Transfer

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Analytical investigations of two-impulse transfers between inclined elliptical orbits, using vector analysis and other mathematical techniques, have yielded pertinent, heretofore unknown facts about an orbital transfer function. A geometric analysis helped to show not only that the minimum velocity increment solution between two points on elliptical orbits could be along a hyperbola, but also that there could be two relative minima in this impulse function. Particular examples of both of these phenomena are given. An eighth-order polynomial expression, the real roots of which may refer to extrema in the impulse function, is then determined. Since it can be shown that some of these roots are extraneous, not corresponding to impulse minima, two test functions are next determined which define regions in which all extrema must lie. These regions identify those roots that do correspond to extrema in the impulse function and those that are extraneous. These new analytical findings have been incorporated into an earlier computer contour mapping program that locates the optimum transfer between elliptical orbits.

Nomenclature

a = semimajor axis
e = eccentricity
i = inclination
Ω = right ascension of ascending node
ω = argument of perigee, angle from reference axis to perigee point
p = semilatus rectum
$\Delta\theta$ = true anomaly angle traversed in transfer orbit plane
μ = gravitation constant
ϕ_1 = angle from reference axis to departure position in initial orbit
ϕ_2 = angle from reference axis to arrival position in terminal orbit
α = angle between $\mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{r}_1$
β = angle between $\mathbf{r}_1$ and $\mathbf{r}_1 - \mathbf{r}_2$
z = term defined in Eq. (7)
Ψ_1 = functional form of 1st velocity increment
Ψ_2 = functional form of 2nd velocity increment
I = functional representation of impulse
I^* = function defined by Eq. (33), whose extrema are also located in Eq. (29)
f = one of test functions used in analyzing short transfer
g = the other test function used in analyzing short transfer
A-H = coefficients that determine interval-finding polynomials
$\Phi_1 - \Phi_9$ = coefficients that determine minimizing polynomial
σ = function defined by Eq. (43)
τ = function defined by Eq. (44)
$\mathbf{e}$ = orbit shape and orientation vector
$\mathbf{r}_1$ = vector from reference position to point of departure on initial orbit
$\mathbf{r}_2$ = vector from reference position to point of arrival on final orbit
$\mathbf{W}$ = unit vector directed along orbit's angular momentum vector
$\mathbf{v}$ = vector defined by Eq. (6)
$\mathbf{V}_{tj}$ = velocity vectors in transfer orbit
$\mathbf{V}_j$ = velocity vectors in initial and final orbit
$\mathbf{V}_{par}$ = velocity vector in parabolic orbit
$\mathbf{V}_r$ = velocity vector in circular orbit
$\mathbf{U}_j$ = unit vectors in direction of radius vectors
$\mathbf{N}, \mathbf{M}, \mathbf{W}_2$ = unit vectors in Cartesian coordinates defining the reference plane

I. Introduction

ONE of the major problems of the nascent space age is concerned with changing orbits in space. The transfer from orbit to orbit can require immense quantities of fuel far beyond the limitations of today's engineering. It is, therefore, of extreme practical interest to be able to locate particular modes of transfer between these orbits that use the least possible fuel.

The most general problem of optimum two-impulse orbital transfer, in which the chief assumption is that the elliptical orbits are unperturbed, permits both the departure point and the arrival point to be arbitrary and finds the single best mode of transfer between the two given orbits. The most general constraint is to fix the end points; then the optimization procedure is carried out solely along a parameter defining all the transfer orbits that go through these given terminals.

The impulsive case of orbital transfer is, of course, an ideal situation. There is one instantaneous thrust from the initial orbit into the transfer orbit; there is a second instantaneous thrust to get into the final orbit. The information gained from the solution of this problem should provide a basis for the study of orbital transfer with finite thrust.

Many technical papers during the last five years have considered the problem of optimizing impulsive orbital transfers. Much of the first work in this field was done by Lawden,[1] but since that time numerous others, including Ting,[2] Horner,[3] Altman-Pistiner,[4, 5] and McCue[6] and Battin,[10] have considered various outgrowths of Lawden's original problem. The analysis presented in this paper is a new treatment of the fixed endpoint problem as stated by Altman-Pistiner and is intended to point out some salient facts that affect McCue's work. This new analysis is easily incorporated into McCue's computer program that uses a contour mapping technique (by repeated solution of the fixed endpoint problem) to optimize orbital transfers between any elliptical orbits. This paper represents a condensation of the fundamental ideas discussed in greater detail by the author in an internal report.[7]

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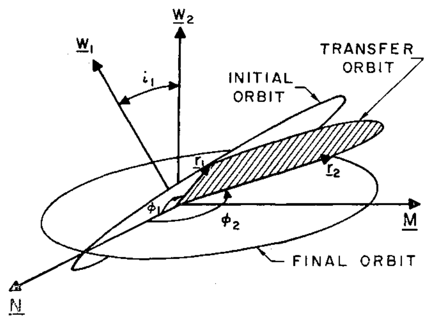

Fig. 1 Transfer geometry.

II. Statement of the Problem

Two Keplerian elliptical orbits in space can be defined by their orbital elements a, e, i, Ω, and ω. In the general two-impulse orbital transfer problem, it is desirable to locate the minimum velocity increment solution between any two such Keplerian orbits. If the plane of the second orbit of the transfer is the reference plane, then i_2, the inclination of the second orbit, is zero. The terms ordinarily referred to as the "nodal" parameters, Ω_1, and Ω_2, are made zero by selecting the line of intersection of the two orbit planes as the reference direction (see Fig. 1):

$$[\mathbf{N} = \mathbf{W}_2 \times \mathbf{W}_1/|\mathbf{W}_2 \times \mathbf{W}_1|] \qquad (1)$$

This leaves seven orbital elements (a_1, e_1, i_1, ω_1 and a_2, e_2, ω_2 with subscripts one and two refering to elements in the first and second orbits, respectively) that define the two orbits between which the transfer is to be accomplished.

Three variables that define all possible means of transferring from the first orbit to the second orbit are ϕ_1, the angle from reference line (**N**) to a departure point on the first orbit; ϕ_2, the angle from reference line to arrival point on second orbit; and p, the semilatus rectum of the transfer orbit between the two points. The parameter p is chosen as the third variable because it simplifies the nature of the impulse function. Other formulations for the third variable can produce serious discontinuities.[6]

The "total impulse" used in transferring between the orbits is defined as the sum of the magnitudes of the velocity changes necessary to get from the first orbit into the transfer orbit and then from the transfer orbit into the second orbit. In this paper, an optimum impulse solution refers to a particular configuration of the three variables which leads to the least possible impulse between two orbits. A minimum impulse solution refers to the transfer orbit that gives the least total impulse for a given arrival-point, departure-point configuration.

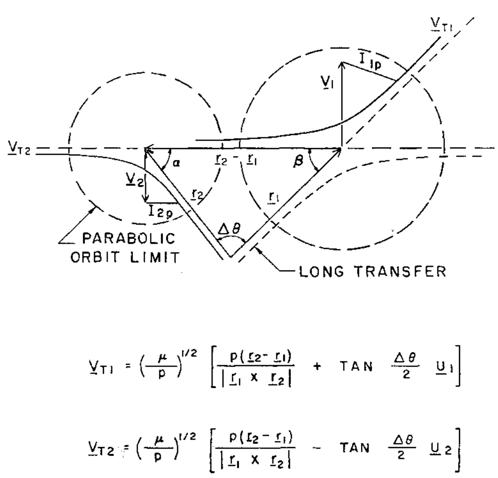

Fig. 2 Stark technique.

III. Transfer Geometry

The orbital transfer problem is most amenable to analysis if the important quantities are expressed in vector formulation. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be vectors from the attracting body to the departure and arrival points; $\mathbf{U}_1$ and $\mathbf{U}_2$ are unit vectors in the direction of $\mathbf{r}_1$ and $\mathbf{r}_2$. Consider further the unit vectors $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{W}_t$. These vectors are normal to the initial, final, and transfer orbit planes and will prove useful in this analysis. To complete the vector description, define the shape and orientation vectors[8] $\mathbf{e}_1$ and $\mathbf{e}_2$ whose magnitudes are equal to orbit eccentricities and whose directions are toward orbit perigees. All quantities encountered in this analysis are expressible in terms of these vectors (see Ref. 6 for more detail).

The part of the transfer orbit traversed is a certain true anomaly interval $\Delta\theta$. This interval may be quickly determined from

$$\cos\Delta\theta = (\mathbf{U}_1 \cdot \mathbf{U}_2) \qquad 0° < \Delta\theta < 180° \qquad (2)$$

No generality is lost if the true anomaly interval is limited to the first two quadrants. Although this does restrict the problem to "short transfers," if the signs of the velocity vectors in the transfer orbit are changed, the long transfers may be considered. The singularities in the impulse function at $\Delta\theta = 180°$ and $\Delta\theta = 0°$ indicate that the problem is simplified by considering the long and short transfers separately. Thus, in order to determine the absolute optimum transfer between two elliptical orbits, it is necessary to compare the optima found from all the short transfers and all the long transfers.

For every elliptical transfer orbit between a given departure point and arrival point, there exists both a short transfer and a long transfer. However, when considering particular hyperbolic transfer orbits, it is important to realize that either the short transfer or the long transfer is meaningless; it would require going out to infinity and back.

IV. Geometric Analysis

A technique developed by Stark[9] refers to a geometrical method of analyzing the two-impulse orbital transfer problem. The fundamental idea of the method, i.e., that all possible transfers between fixed terminals on any elliptical orbits can be represented by two hyperbolas, provided the stimulus for much of this analysis.

For any two elliptical orbits, let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the vectors from the reference position on the line of intersection to the departure and arrival points, respectively. The angle between them is $\Delta\theta$, and the size of this angle can be selected to be always in the first two quadrants without any loss of generality. By forming the vector $\mathbf{r}_2 - \mathbf{r}_1$, a triangle is made of the three vectors in the transfer orbit plane.

Define the two angles α and β (Fig. 2) as follows:

$$\beta = \arcsin|\mathbf{r}_2| \sin\Delta\theta/|\mathbf{r}_2 - \mathbf{r}_1| \qquad (3)$$

$$\alpha = \pi - (\beta + \Delta\theta) \qquad (4)$$

Consider the locus of all possible velocity vectors that can act upon the point defined by $\mathbf{r}_1$, and trace a conical orbit path that goes through the point defined by $\mathbf{r}_2$. This locus defines all possible conic transfer orbits between the two points, since a particular orbit is uniquely defined by its velocity vector at a given position.

The velocity vector of any transfer orbit at the particular point $\mathbf{r}_1$ is given by (see detailed derivation in Ref. 7)

$$\mathbf{V}_{t1} = \mathbf{v} + z\mathbf{U}_1 \qquad (5)$$

where

$$\mathbf{v} = (\mu p)^{1/2}(\mathbf{r}_2 - \mathbf{r}_1)/|\mathbf{r}_1 \times \mathbf{r}_2| \qquad (6)$$

$$z = (\mu/p)^{1/2} \tan\Delta\theta/2 \qquad (7)$$

where p is the semilatus rectum of the transfer orbit. Then $\mathbf{V}_{t1}$ may be written as a function of this variable p:

$$\mathbf{V}_{t1}(p) = \left(\frac{\mu}{p}\right)^{1/2}\left[\frac{p|\mathbf{r}_2 - \mathbf{r}_1|}{|\mathbf{r}_1 \times \mathbf{r}_2|}\,\mathbf{m} + \tan\frac{\Delta\theta}{2}\,\mathbf{U}_1\right] \qquad (8)$$

where $\mathbf{m}$ is unit vector in the direction of $\mathbf{r}_2 - \mathbf{r}_1$. Every positive value of p greater than zero defines a certain transfer orbit whose velocity vector at $\mathbf{r}_1$ has components in the direction of $\mathbf{m}$ and $\mathbf{U}_1$.

For any coordinate axes, the locus of all points such that the product of the coordinates is a constant forms a hyperbola with the axes as asymptotes. Since the product of the magnitudes of the components in the $\mathbf{m}$ direction and the $\mathbf{U}_1$ direction is independent of p,

$$\left[\frac{(\mu p)^{1/2}|\mathbf{r}_2 - \mathbf{r}_1|}{|\mathbf{r}_1 \times \mathbf{r}_2|}\right]\left[\left(\frac{\mu}{p}\right)^{1/2}\tan\frac{\Delta\theta}{2}\right] = \frac{\mu\tan(\Delta\theta/2)|\mathbf{r}_2 - \mathbf{r}_1|}{|\mathbf{r}_1 \times \mathbf{r}_2|} \qquad (9)$$

the formulation of $\mathbf{V}_{t1}$ defines a hyperbola with the oblique coordinates established by $\mathbf{m}$ and $\mathbf{U}_1$ as asymptotes. Thus the locus of all possible velocity vectors leaving $\mathbf{r}_1$ and arriving at $\mathbf{r}_2$ on a conic path forms a hyperbola.

Similarly, at $\mathbf{r}_2$, the velocity vector for any transfer orbit (dependent upon its semilatus rectum) is given by

$$\mathbf{V}_{t2} = \mathbf{v} - z\mathbf{U}_2 \qquad (10)$$

This defines another hyperbola that represents the locus of all possible transfer orbits leaving from $\mathbf{r}_1$ and arriving at $\mathbf{r}_2$. These are shown in Fig. 2. It is important to note that, for every p, there is one point on each of these hyperbolas that represents the transfer orbit.

These two hyperbolas refer to the so-called short transfer in which the true anomaly interval traversed in the transfer orbit is less than 180°. If the true anomaly interval is greater than 180° ("long transfer"), the other branches of these same two hyperbolas represent the locus of all transfer orbits. These are obtained by simply changing the sign of $\mathbf{V}_{t1}$ and $\mathbf{V}_{t2}$.

In Fig. 2, the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$, defining the initial and final orbits, are in the transfer orbit plane to simplify the analysis. This diagram represents a coplanar transfer and $\mathbf{V}_1$ and $\mathbf{V}_2$, defined by

$$\mathbf{V}_1 = (\mu/p_1)^{1/2}\mathbf{W}_1 \times (\mathbf{e}_1 + \mathbf{U}_1) \qquad (11)$$

$$\mathbf{V}_2 = (\mu/p_2)^{1/2}\mathbf{W}_2 \times (\mathbf{e}_2 + \mathbf{U}_2) \qquad (12)$$

which must have magnitudes less than parabolic speed (V_{par}):

$$V_{\text{par}}^2 = 2\mu/r \qquad (13)$$

In the diagram, the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ (which uniquely define the initial and final orbits) emanate from $\mathbf{r}_1$ and $\mathbf{r}_2$ and must lie within a certain radius containing all elliptical orbits.

In finding the minimum velocity change solution for this two-impulse case, the function to be minimized is

$$I(p) = \Psi_1(p) + \Psi_2(p) \qquad (14)$$

where

$$\Psi_1(p) = |\pm\mathbf{V}_{t1}(p) - \mathbf{V}_1| \qquad (15)$$

$$\Psi_2(p) = |\mathbf{V}_2 \mp \mathbf{V}_{t2}(p)| \qquad (16)$$

The double sign on the transfer velocity vector refers to short and long transfers (upper sign is short). In the diagram, this optimization procedure requires that the sum of the distances from $\mathbf{V}_1$ and $\mathbf{V}_2$ to their respective transfer loci be minimized. For every p, there is one and only one point on each hyperbola corresponding to that transfer orbit. The distances marked I_{1p} and I_{2p} (in Fig. 2) represent simply a particular transfer orbit chosen for illustrative purposes.

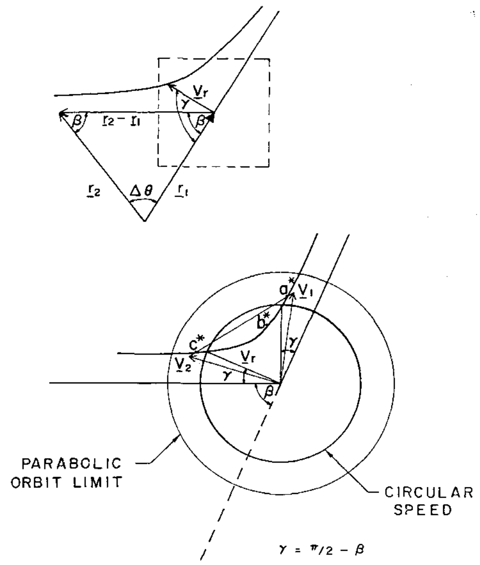

Fig. 3 Double minimum (case of equal radii).

Their sum represents the impulse necessary to transfer between these two points along that particular conic.

The geometric technique provides a picture of what is occurring in the two-impulse orbital transfer. By comparing the magnitudes of the impulse vectors for different transfer orbits, one can gain an intuitive feeling for the size of the impulse for a particular transfer orbit. More important, though, was the fact that this geometric technique offered clues to two of the more important questions in the field.

Recently, McCue[6] conducted a numerical search for the minimum impulse for each arrival-point, departure-point configuration and then, by a method of contour mapping, located the optimum transfer between any two elliptical orbits. One of his early assumptions was that there could only be one minimum in the impulse function (variable p, semilatus rectum of transfer orbit) for a fixed pair of terminals. The geometric technique described in Sec. IV clearly shows the existence of other configurations under which a double minimum may be present.

It has been implied in nearly all of the definitive analytical works in this area, such as those by Altman and Pistiner,[4, 5] that the minimum velocity increment solution between points on elliptical orbits was always an ellipse. The geometric technique suggested the existence of hyperbolic minima for certain configurations; this fact was subsequently proved.

V. Location of Double Minimum

In order to assert that there can be a double minimum in the impulse function for fixed terminals, it is necessary only to find an example. By considering a particular case with unique symmetry properties, this example can be readily illustrated.

Consider the case where $|\mathbf{r}_1| = |\mathbf{r}_2|$ (Fig. 3). This makes the angle α (Fig. 2) equal to the angle β. Then the hyperbolas formed between the oblique axes at both the departure point and the arrival point are equivalent. If the entire coordinate system at $\mathbf{r}_2$ were flipped over and translated to $\mathbf{r}_1$, then these two hyperbolas would become coincident; they would match up point for point, transfer orbit for transfer orbit. Then the impulse function for particular elliptical orbits (defined by $\mathbf{V}_1$ and $\mathbf{V}_2$, both of which now act at the same point) is only the sum of the distances from $\mathbf{V}_1$ and $\mathbf{V}_2$ to all points on the hyperbola. Then, for this case, the minimum impulse solution corresponds to the point on the hyperbola from which the sum of the distances to $\mathbf{V}_1$ and $\mathbf{V}_2$ is a minimum.

For points with equal radii, one possible transfer orbit corresponds to a circular transfer. This transfer has a ve-

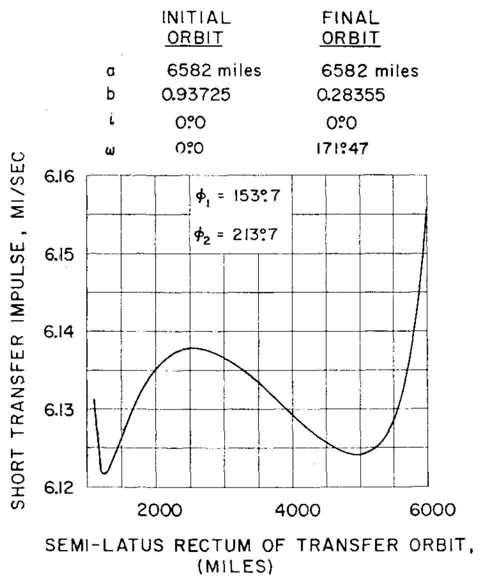

Fig. 4 Semilatus rectum of transfer orbit, miles.

locity vector (V_r) perpendicular to the radius vector, and its magnitude is given by

$$V_r^2 = \mu/r \tag{17}$$

All velocity vectors emanating from r_1 which have magnitudes less than $(2\mu/r)^{1/2}$ define elliptical initial and final orbits. In the diagram, this range for V_1 and V_2 is described by a circle marked parabolic orbit limit.

Suppose V_1 and V_2 are located in such positions (see Fig. 3), relative to each other, that the line connecting them intersects the hyperbola (either short transfer branch or long transfer branch) twice. As p varies from zero to its unbounded upper value, all possible transfer orbits have a corresponding point on the hyperbola. As p increases along the hyperbola, the value of the impulse is obviously decreasing until p reaches the value corresponding to a^*, where the line between V_1 and V_2 intersects the hyperbola. For values of p slightly larger than a^* (such as the p corresponding to point b^*), according to the triangle inequality, the impulse must be higher. Thus the value of p at a^* must constitute a relative minimum in the impulse function.

As p nears the value corresponding to point c^* on the hyperbola, the triangle inequality states that the necessary transfer impulse is going down again. For points past c^*, the impulse is rising again, and thus c^* must also be a relative minimum. The fact that there can be two minima is thus demonstrated.

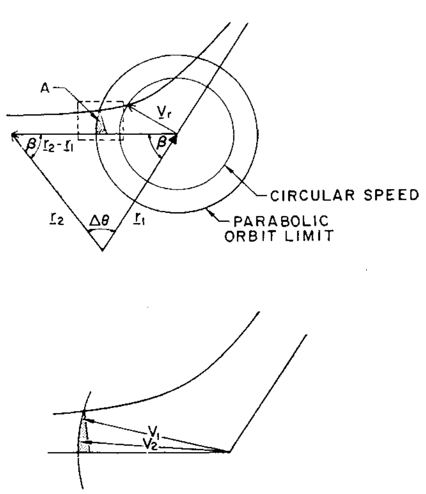

Fig. 5 Example of hyperbolic minimum case of equal radii.

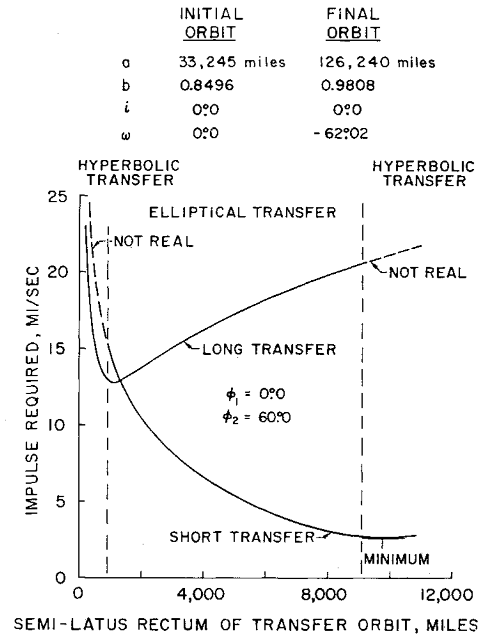

Fig. 6 Semilatus rectum of transfer orbit, miles.

A numerical example was computed, and indeed a double minimum (see Fig. 4) occurred. The orbital elements for that particular fixed terminal case are given on the graph. For this case, the long transfer provides a greater impulse requirement for all transfers; thus only the short transfer is plotted.

VI. Location of Hyperbolic Minimum

The assumption has been made, in prior two-impulse orbital transfer studies, that the minimum transfer between two points on elliptical orbits always lies along an ellipse. Although this has never been proved, it has been generally accepted. Use of the geometric technique showed this assumption to be false.

For the case of $|r_1| = |r_2|$, it is clear from Fig. 5 that a hyperbolic minimum may exist. Once again, the coordinate system at r_2 is rotated and flipped such that all possible transfer orbits are given by one hyperbola. If the vectors V_1 and V_2 lie in the shaded region (see insert), the shortest distance from each to the transfer orbit hyperbola arrives at a point on that hyperbola outside the parabolic orbit limit. Since the least velocity increment, both to arrive in the transfer orbit and depart from it, lies along hyperbolic transfers, the sum of the two (the impulse) must have its minimum along a hyperbolic orbit.

In investigating the more general case of nonequal radii, the geometry yielded not only configurations for which the minimum velocity increment solution could lie along a hyperbola, but also some other interesting properties about this orbital transfer function. A detailed discussion of these analytically cogent properties, which extend one's intuitive understanding of the basic problem, is omitted here but can be found in Ref. 7. This further analysis, more general than the case of the equal radii, pointed out larger regions in which a hyperbolic transfer could be a minimum. Data from this study was used to prepare Fig. 6, an actual configuration whose hyperbolic minimum was predicted by the geometric technique, thus demonstrating its efficacy.

VII. Analysis of Impulse Function

The location of these peculiarities in the impulse function prompted an analytic search into the equations that describe the impulse problem. New analytic boundaries, different from the parabolic orbit limits, were sought for the minima. For fixed terminals (once again it should be pointed out that this is a restricted case of the more general problem of optimizing between any points on elliptical orbits), the impulse function is only dependent on p, the semilatus rectum of the transfer orbit. This impulse function, defined by Eq. (14), has an extremum at all points where

$$\frac{\partial I}{\partial p} = \frac{\partial \Psi_1}{\partial p} + \frac{\partial \Psi_2}{\partial p} = 0 \tag{18}$$

In the analysis of the impulse function carried out here, only the short transfers are considered. It is shown in a subsequent section that the extension to include the long transfers is very simple. Now

$$\begin{aligned}\psi_1(p) &= [(\mathbf{V}_{t1}(p) - \mathbf{V}_1)\cdot(\mathbf{V}_{t1}(p) - \mathbf{V}_1)]^{1/2} \\ &= [(\mathbf{v}(p) + z(p)\mathbf{U}_1 - \mathbf{V}_1)\cdot(v(p) + z(p)\mathbf{U}_1 - \mathbf{V}_1)]^{1/2} \\ &= [f(p)]^{1/2}\end{aligned} \tag{19}$$

where

$$\begin{aligned}f(p) &= \mathbf{v}(p)\cdot\mathbf{v}(p) + z^2(p) + \mathbf{V}_1\cdot V_1 - \\ &\quad 2z(p)\mathbf{V}_1\cdot\mathbf{U}_1 - 2\mathbf{V}_1\cdot\mathbf{v}(p) + 2z(p)\mathbf{v}(p)\cdot\mathbf{U}_1 \quad (20)\\ &= Ap + 2Bp^{1/2} + G - 2Cp^{-1/2} - Dp^{-1}\end{aligned}$$

where the coefficients are given in Table 1.

Similarly,

$$\Psi_2(p) = [g(p)]^{1/2} \tag{21}$$

where

$$g(p) = Ap + 2Ep^{1/2} + H - 2Fp^{-1/2} - DP^{-1} \tag{22}$$

where the new coefficients are also given in Table 1. Then, in order for impulse to be an extremum,

$$\frac{\partial \Psi_1}{\partial p} + \frac{\partial \Psi_2}{\partial p} = \frac{1}{2\Psi_1}\frac{\partial f}{\partial p} + \frac{1}{2\Psi_2}\frac{\partial g}{\partial p} = 0 \tag{23}$$

$$\rightarrow \quad \frac{\Psi_1(p)}{\Psi_2(p)} = -\frac{\partial f/\partial p}{\partial g/\partial p} \tag{24}$$

Since $\Psi_1(p)$ and $\Psi_2(p)$ are always positive, it is easy to see from Eq. (24) that $\partial f/\partial p$ and $\partial g/\partial p$ must be of different sign before an extremum can occur in the impulse function. This important fact permits the identification of the extraneous roots in the eigth-order polynomial that will be derived. Then

$$\partial f/\partial p = A + Bp^{-1/2} + Cp^{-3/2} + Dp^{-2} \tag{25}$$

$$\partial g/\partial p = A + Ep^{-1/2} + Fp^{-3/2} + Dp^{-2} \tag{26}$$

Before a meaningful expression can be worked out for the extrema in the impulse, Eq. (24) must be squared. Then the necessary expression becomes

$$\frac{f(p)}{g(p)} = \frac{(\partial f/\partial p)^2}{(\partial g/\partial p)^2}$$

or

$$f(p)\left(\frac{\partial g}{\partial p}\right)^2 - g(p)\left(\frac{\partial f}{\partial p}\right)^2 = 0 \tag{27}$$

When this equation is multiplied out using Eqs. (19, 20, 25, and 26), together with the substitution

$$s = p^{1/2} \tag{28}$$

the necessary condition for an extremum becomes

$$\Phi_1 s^8 + \Phi_2 s^7 + \Phi_3 s^6 + \Phi_4 s^5 + \Phi_5 s^4 + \Phi_6 s^3 + \Phi_7 s^2 + \Phi_8 s + \Phi_9 = 0 \tag{29}$$

where the coefficients Φ_i, i = 1–8, are given in Table 2. The real roots of this eighth-order polynomial must include all the values of p for which the impulse is an extremum.

The squaring process introduced in Eq. (27) added some extraneous roots to the octic, i.e., roots that do not correspond to extrema in $I(p)$. These can be identified by factoring Eq. (27) as the difference of two squares:

$$f(p)\left(\frac{\partial g}{\partial p}\right)^2 - g(p)\left(\frac{\partial f}{\partial p}\right)^2 = 0$$

$$\rightarrow \left(\Psi_1(p)\frac{\partial g}{\partial p} + \Psi_2(p)\frac{\partial f}{\partial p}\right)\left(\Psi_1(p)\frac{\partial g}{\partial p} - \Psi_2(p)\frac{\partial f}{\partial p}\right) = 0 \tag{30}$$

Since $\partial f/\partial p$ and $\partial g/\partial p$ must be of different sign, only those real values of p which are roots of

$$\Psi_1(p)\frac{\partial g}{\partial p} + \Psi_2(p)\frac{\partial f}{\partial p} = 0 \tag{31}$$

are true extrema of $I(p)$. It is easily shown[7] that the equation

$$\Psi_1(p)\frac{\partial g}{\partial p} - \Psi_2(p)\frac{\partial f}{\partial p} = 0 \tag{32}$$

contains the extraneous roots of the octic and refers to extrema in another function, $I^*(p)$, where

$$I^*(p) = \Psi_1(p) - \Psi_2(p) \tag{33}$$

Inquiries into the nature of this octic suggest that four of these roots refer to extrema in $I^*(p)$. Although no general proof has been made, if this fact were true for all configurations, then there could be no more than two minima on either transfer branch. This would greatly simplify the application of the contour mapping approach.

Table 1 Coefficients of test functions

A	=	$\mu\dfrac{\lvert\mathbf{r}_2 - \mathbf{r}_1\rvert^2}{\lvert r_1 \times \mathbf{r}_2\rvert^2}$
B	=	$-\dfrac{\mu}{(p_1)^{1/2}\lvert\mathbf{r}_1 \times \mathbf{r}_2\rvert}\{(\mathbf{r}_2 - \mathbf{r}_1)\cdot[\mathbf{W}_1 \times (\mathbf{e}_1 + \mathbf{U}_1)]\}$
C	=	$\dfrac{\mu\tan(\Delta\theta/2)}{(p_1)^{1/2}}[\mathbf{U}_1\cdot(\mathbf{W}_1 \times \mathbf{e}_1)]$
D	=	$-\mu\tan^2\dfrac{\Delta\theta}{2}$
E	=	$\dfrac{\mu}{(p_2)^{1/2}\lvert\mathbf{r}_1 \times \mathbf{r}_2\rvert}\{(\mathbf{r}_2 - \mathbf{r}_1)\cdot[\mathbf{W}_2 \times (\mathbf{e}_2 + \mathbf{U}_2)]\}$
F	=	$\dfrac{-\mu\tan(\Delta\theta/2)}{(p_2)^{1/2}}[\mathbf{U}_2\cdot(\mathbf{W}_2 \times \mathbf{e}_2)]$
G	=	$\dfrac{\mu}{p_1}[\mathbf{W}_1 \times (\mathbf{e}_1 + \mathbf{U}_1)]^2 + \dfrac{2\mu\tan(\Delta\theta/2)}{\lvert\mathbf{r}_1 \times \mathbf{r}_2\rvert}[\mathbf{U}_1\cdot(\mathbf{r}_2 - \mathbf{r}_1)]$
H	=	$\dfrac{\mu}{p_2}[\mathbf{W}_2 \times (\mathbf{e}_2 + \mathbf{U}_2)]^2 - \dfrac{2\mu\tan(\Delta\theta/2)}{\lvert\mathbf{r}_1 \times \mathbf{r}_2\rvert}[\mathbf{U}_2\cdot(\mathbf{r}_2 - \mathbf{r}_1)]$

Table 2 Coefficients of eighth-order polynomial

Φ_1	=	$A^2(G - H) + A(E^2 - B^2)$
Φ_2	=	$A^2(4F - 4C) + A(2EG - 2BH) + 2E^2B - 2EB^2$
Φ_3	=	$A(8BF - 8EC + 2EF - 2BC) + E^2G - HB^2$
Φ_4	=	$A(4BD - 4ED + 2FG - 2CH) + 4BEF - 2CE^2 - 4BCE + 2FB^2$
Φ_5	=	$D(2AG - 2HA - E^2 + B^2) + A(F^2 - C^2) - 2BCH + 2GEF$
Φ_6	=	$D(4FA - 4AC + 2EG - 2BH) + 4FBC - 4CEF + 2BF^2 - 2EC^2$
Φ_7	=	$D(8BF - 8EC + 2BC - 2EF) + F^2G - C^2H$
Φ_8	=	$D^2(4B - 4E) + D(2FB - 2HC) + 2FC^2 - 2CF^2$
Φ_9	=	$D^2(G - H) + D(C^2 - F^2)$

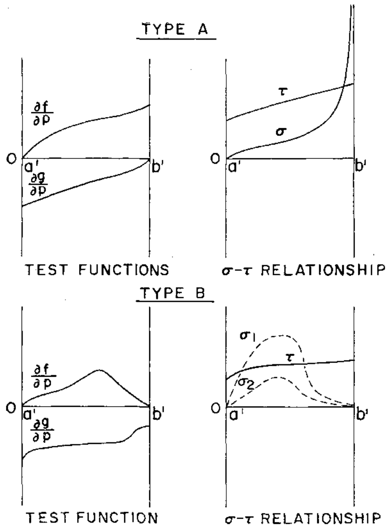

Fig. 7 Illustration of intervals.

Altman and Pistiner[5] have recently derived, using a different method, a similar eighth-order polynomial. However, they neglect the consideration of extraneous roots. It can be seen from the foregoing discussion that some extraneous roots to this octic must exist.

VIII. The Boundaries on Minima

Since a necessary condition for the existence of an extremum in the impulse function is that $\partial f/\partial p$ and $\partial g/\partial p$ be of different sign, analyses were next directed to determine the values for p for which they could be of different sign.

From Eqs. (25) and (26),

$$\lim_{p\to\infty} \frac{\partial f}{\partial p} = \lim_{p\to\infty} \frac{\partial g}{\partial p} = A \tag{34}$$

where

$$A = \frac{\mu|\mathbf{r}_2 - \mathbf{r}_1|^2}{|\mathbf{r}_1 \times \mathbf{r}_2|^2} > 0 \tag{35}$$

and

$$\lim_{p\to 0^+} \frac{\partial f}{\partial p} = \lim_{p\to 0^+} \frac{\partial g}{\partial p} = -\infty \tag{36}$$

because

$$D = -\mu \tan^2\frac{\Delta\theta}{2} \tag{37}$$

Since for p both very small and very large, $\partial f/\partial p$ and $\partial g/\partial p$ have the same sign, we know that the region in which $\partial f/\partial p$ and $\partial g/\partial p$ are of different sign may definitely be bounded. The boundaries in which all minima in the impulse function (on short transfer side) must lie are given by the least positive value of p and the greatest positive value of p at which either

$$\partial f/\partial p = 0 \quad \text{or} \quad \partial g/\partial p = 0 \tag{38}$$

If $s = p^{1/2}$, then $\partial f/\partial p = 0$ where

$$As^4 + Bs^3 + Cs + D = 0 \tag{39}$$

and $\partial g/\partial p = 0$ where

$$As^4 + Es^3 + Fs + D = 0 \tag{40}$$

These two quartics may be solved easily to obtain the values for s and p that bound the region in which minima may lie. In Ref. 7 it is shown that, because of symmetry, these equations may also be used to give boundaries on the long transfer impulse minima. Thus, definite, analytic boundaries on the possible range of the impulse minima have been ascertained.

IX. Intervals

Since both $\partial f/\partial p$ and $\partial g/\partial p$ have negative values for p very small and positive values for p very large, both expressions [(39) and (40)] must have an odd number of positive real roots. Each of these quartic equations may have either one or three real positive roots. Regardless of how many of these roots each of these quartics has, all possible combinations of the roots can be studied by investigating two types of intervals in which $\partial f/\partial p$ and $\partial g/\partial p$ may be of different sign.

Type A: ($\partial f/\partial p$ and $\partial g/\partial p$ of different sign in $[a', b']$)

$$\left(\frac{\partial f}{\partial p}\right)_{p=a'} = 0 \qquad \left(\frac{\partial g}{\partial p}\right)_{p=b'} = 0 \tag{41}$$

Type B: ($\partial f/\partial p$ and $\partial g/\partial p$ of different sign in $[a', b']$)

$$\left(\frac{\partial f}{\partial p}\right)_{p=a'} = 0 \qquad \left(\frac{\partial f}{\partial p}\right)_{p=b'} = 0 \tag{42}$$

It is important to note that, if, in Eq. (41), $\partial f/\partial p$ and $\partial g/\partial p$ are zero at opposite ends of the interval from those given, the problem is not really changed. Similarly, if in Eq. (42) it is $\partial g/\partial p$ that is zero at both ends, the analysis of the types of intervals still holds. In type A each of the functions is zero at one end of the interval; in type B, one function is zero at both ends of the interval. The two types of intervals are illustrated in Fig. 7.

These intervals are divided into two types because the number of minima possible in a given interval is determined by its type. Define two functions $\sigma(p)$ and $\tau(p)$ as

$$\sigma(p) = -(\partial f/\partial p)/(\partial g/\partial p) \tag{43}$$

$$\tau(p) = \Psi_1(p)/\Psi_2(p) \tag{44}$$

Obviously, an extremum in the impulse function occurs for all p at which $\sigma(p) = \tau(p)$.

Consider an interval of type A. τ is monotonic increasing and positive for all p in $[a', b']$. Also, note that

$$\sigma(a) = 0 \tag{45}$$

and

$$\lim_{p\to b'} \sigma(p) = \infty \tag{46}$$

From Fig. 7, it is clear σ and τ must intersect at least one time (producing one extremum) in that interval. If they are equal more than once, they must intersect an odd number of times.

Consider next an interval of type B. Once again τ is monotonic increasing and positive for all p in $[a', b']$. Here, though,

$$\sigma(a') = 0 \tag{47}$$

and

$$\sigma(b') = 0 \tag{48}$$

whereas for all p in $[a', b']$, $\sigma(p) > 0$. It is evident from Fig. 7 that σ and τ must intersect an even number of times in intervals of this type.

All possible permutations of the roots of these quartics can be manipulated to reduce the problem to an analysis of these intervals. Most frequently, both $\partial f/\partial p$ and $\partial g/\partial p$ have one real, positive root and produce an interval of type A in which σ and τ intersect one time. It is also true that, for the majority of the cases, the first and last real positive roots of the two quartics will limit the search for the minimum impulse to elliptical transfer orbits. These analyses do, however, explain the existence of the two peculiarities located earlier. They also form the basis for the modification of McCue's computer programs.[6]

X. Long and Short Transfer

For nearly all the equations derived in the preceding sections, it was assumed that the two-impulse orbital transfer was accomplished with a true anomaly interval in the transfer orbit of less than 180° short transfer. The symmetry of the problem makes extension to include the long transfers very simple. To obtain the absolute minimum impulse, the two are then compared.

Because of the symmetry (see Ref. 7 for detailed derivation of long transfer equations) it can be shown that the real, negative roots of Eqs. (39) and (40) determine intervals on the long transfer side which may produce minima. Similarly, it is the real, negative roots of the general octic [Eq. (29)] which appear within those specified intervals that determine values of p for which the long transfer may be an extremum.

This implies that all the analysis can be conducted by examining three equations, two quartic and one octic. The real roots of these equations, positive for short transfer and negative for long transfer, define all the intervals in which the extrema may exist and then locate the values of p at which extrema actually occur.

XI. Summary

New analytical approaches to the two-impulse orbital transfer problem are developed in this paper. This development precipitated the discovery of both the hyperbolic minimum and the double minimum in the minimum velocity increment solution between points on elliptical orbits. Further analyses produced on eighth-order polynomial, applicable even for inclined orbits, whose roots contain all possible extrema in the impulse function. Next, test functions were located which placed bounds on the regions in which these extrema could exist and identified those roots of the octic which were extraneous. The explanation of these extraneous roots, not corresponding to minima in the impulse function, was given.

All these results have been used to modify an earlier computer program.[6] It is now possible to locate not only the absolute minimum two-impulse transfer between fixed terminals for any elliptical orbit pair, but also the absolute optimum transfer between those orbits.

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